

Maximal superintegrability of Benenti systems

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Abstract

For a class of Hamiltonian systems naturally arising in the modern theory of separation of variables, we establish their maximal superintegrability by explicitly constructing the additional integrals of motion.

For a Hamiltonian dynamical system on a $2n$ -dimensional phase space to be completely integrable, the classical Liouville theorem requires n involutive integrals of motion. Solving the equations of motion then amounts to n quadratures, but performing the latter still can be a challenge. However, if there are $m > n$ integrals of motion, only $2n - m$ quadratures are required. In particular, for the so-called *maximally superintegrable* systems (i.e., those possessing $2n - 1$ integrals of motion) we need just *one* quadrature.

Superintegrable systems on two- and three-dimensional Euclidean spaces and spaces of constant curvature were extensively studied in the literature, see e.g. [1]–[7] and also [8] for the case of nonconstant curvature. On the other hand, there is not much known about superintegrability in higher dimensions. Remarkable exceptions include n -dimensional Winternitz–Smorodinsky model [1] and its generalizations [9], Calogero–Moser–Sutherland system [10, 11] and the systems with isochronic potentials [12] and modified Coulomb potential [13]. Notice that for the majority of these systems the separation of variables takes place in the original coordinates and the Hamiltonians themselves are additively separable.

In this letter we consider the Stäckel systems from the class described by Benenti [14, 15], so the corresponding Hamiltonians are quadratic in momenta. We show that some of these systems (namely those associated with flat or constant curvature metrics) are maximally superintegrable for arbitrary dimension n of the configuration space, and it seems plausible that they are multiseparable too.

Consider an n -dimensional manifold M with local coordinates q^1, \dots, q^n and the $n \times n$ matrices

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \dots & 0 & 0 & 1 & q^1 \\ 0 & \dots & 0 & 1 & q^1 & q^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & q^1 & q^2 & \dots & q^{n-2} \\ 1 & q^1 & q^2 & q^3 & \dots & q^{n-1} \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} -q^1 & 1 & 0 & 0 & \dots & 0 \\ -q^2 & 0 & 1 & 0 & \dots & 0 \\ -q^3 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -q^{n-1} & 0 & 0 & \dots & 0 & 1 \\ -q^n & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

If we interpret G as a contravariant metric on M , then L is a special conformal Killing tensor for G in the sense of Crampin and Sarlet [16] (and moreover, this holds for any metric of the form $L^i G$, $i \in \mathbb{Z}$ [17]), and hence for any integer i the quantities

$$E_{i,r} = \frac{1}{2} \sum_{l,m=1}^n (K_r L^i G)^{lm} p_l p_m, \quad r = 1, \dots, n, \quad (1)$$

are in involution with respect to the canonical Poisson bracket in (p, q) -variables: $\{q^i, p_j\} = \delta_j^i$, $i, j = 1, \dots, n$. Here $(q^1, \dots, q^n, p_1, \dots, p_n)$ are local coordinates on the cotangent bundle T^*M and the Killing tensors K_r are constructed from L as follows [14]:

$$K_1 = \mathbb{I}, \quad K_{r+1} = \sum_{k=0}^r q^k L^{r-k}, \quad r = 1, \dots, n-1,$$

where we set for convenience $q^0 \equiv 1$ and \mathbb{I} stands for the $n \times n$ unit matrix. Notice that q^r are coefficients of the characteristic polynomial of L :

$$\det(\xi I - L) = \xi^n + q^1 \xi^{n-1} + \dots + q^n. \quad (2)$$

The Hamiltonians $E_{i,r}$ belong to the family of Stäckel separable systems, and the separation of variables is achieved by passing to the coordinates (μ, λ) related to (p, q) by the formulas

$$p_i = - \sum_{k=1}^n (\lambda^k)^{n-i} \mu_k / \Delta_k, \quad q^i = (-1)^i \sigma_i(\lambda), \quad i = 1, \dots, n,$$

where σ_i is an i^{th} symmetric polynomial in the variables $\lambda^1, \dots, \lambda^n$ ($\sigma_0 = 1$, $\sigma_1 = \sum_{i=1}^n \lambda^i$, \dots , $\sigma_n = \lambda^1 \lambda^2 \dots \lambda^n$) and $\Delta_i = \prod_{j=1, j \neq i}^n (\lambda^i - \lambda^j)$.

The main advantage of using the ‘nonstandard’ variables (p, q) is that $E_{i,r}$ are polynomial in q ’s for $i \geq 0$, so it is natural to search for additional integrals of motion that also are polynomial in p ’s and q ’s.

For $i = 0, \dots, n$ the (contravariant) metrics $(G_i)^{rs} \equiv (L^i G)^{rs}$ are flat and we have

$$E_{i,1} = \frac{1}{2} \sum_{k=0}^{n-i-1} q^k \sum_{j=k+1}^{n-i} p_j p_{n-i+k-j+1} - \frac{1}{2} \sum_{k=1}^i q^{n-i+k} \sum_{j=1}^k p_{n-i+j} p_{n-i+k-j+1}.$$

The metric G_{n+1} has nonzero constant curvature and we obtain

$$E_{n+1,1} = \frac{1}{2} \sum_{i,j=1}^n (q^i q^j - q^{i+j}) p_i p_j,$$

where we assume that $q^k \equiv 0$ for $k > n$.

We readily find that

$$\begin{aligned} \{E_{i,1}, p_{n-i}\} &= 0, \quad i = 0, \dots, n-1, \\ \{E_{i,1}, q^n p_{n-i+2}\} &= 0, \quad i = 2, \dots, n+1, \\ \{E_{i,1}, q^1\} &= -p_{n-i}, \quad i = 0, \dots, n-1. \end{aligned}$$

Thus q^{n-i} is a cyclic coordinate for $E_{i,1}$, $i = 0, \dots, n-1$, and $F_{i,r} = \{E_{i,r}, p_{n-i}\}$ for $r = 2, \dots, n$ provide $(n-1)$ additional, quadratic in momenta, integrals for the dynamical systems associated with the Hamiltonians $E_{i,1}$, $i = 0, \dots, n-1$.

Consider s functions f_i of the form

$$f_i = \sum_{j,k=1}^n A_i^{jk}(q) p_j p_k + C_i(q).$$

It is easily seen that if the matrices A_i are linearly independent over the field of all locally analytic functions of $q^1, \dots, q^n, p_1, \dots, p_n$, then f_i are *functionally independent*.

Using this result we readily find that $(2n-1)$ quantities $E_{i,s}$, $s = 1, \dots, n$, and $F_{i,r}$, $r = 2, \dots, n$ are functionally independent for any $i = 0, \dots, n-1$, and hence the Hamiltonians $E_{i,1}$ are maximally

superintegrable for all these i . The same is true for $i = n$ and $i = n + 1$, but in this case the additional integrals have the form $F_{i,r} = \{E_{i,r}, q^n p_{n-i+2}\}$, $r = 2, \dots, n$, and they again are quadratic in momenta.

This picture can be extended to the case of nonzero potentials. Namely, consider the basic separable potentials given by the recursion relations [18, 19]

$$\begin{aligned} V_r^{(m+1)} &= V_{r+1}^{(m)} + V_r^{(1)} V_1^{(m)}, \quad m = 1, 2, \dots, \quad V_r^{(1)} = -q^r, \\ V_r^{(0)} &= 0, \\ V_r^{(-m-1)} &= V_{r-1}^{(-m)} + V_r^{(-1)} V_n^{(-m)}, \quad m = 1, 2, \dots, \quad V_r^{(-1)} = -q^{r-1}/q^n. \end{aligned} \quad (3)$$

As the potentials $V_1^{(m)}$ are independent of q^j for $m = j - n, \dots, j - 1$, the above analysis can be generalized to yield the following result:

Theorem 1 *For any natural $n \geq 2$ the dynamical systems with the Hamiltonians $H_{i,1}^{(k)} = E_{i,1} + V_1^{(k)}$ are maximally superintegrable for all $i = 0, \dots, n-1$, $k = -i, \dots, n-1-i$ and $i = n, n+1$, $k = 2-i, \dots, n+1-i$. Namely, besides n involutive integrals $H_{i,r}^{(k)} = E_{i,r} + V_r^{(k)}$, $r = 1, \dots, n$, the dynamical system associated with $H_{i,1}^{(k)}$ has $(n-1)$ additional integrals $F_{i,s}^{(k)}$, $s = 2, \dots, n$, of the following form:*

- a) $F_{i,s}^{(k)} = \{H_{i,s}^{(k)}, p_{n-i}\}$, $s = 2, \dots, n$, for $i = 0, \dots, n-1$, $k = -i, \dots, n-1-i$,
- b) $F_{i,s}^{(k)} = \{H_{i,s}^{(k)}, q^n p_{n-i+2}\}$ for $i = n, n+1$, $k = 2-i, \dots, n+1-i$,

and these $(2n-1)$ integrals $(H_{i,r}^{(k)}, r = 1, \dots, n, \text{ and } F_{i,s}^{(k)}, s = 2, \dots, n)$ are functionally independent.

Let us illustrate this theorem by a nontrivial example. First, let $n = 4$, $i = 4$, $k = -2$. The functions

$$\begin{aligned} H_{4,1}^{(-2)} &= -\frac{1}{2}q^1 p_1^2 - q^2 p_1 p_2 - q^3 p_1 p_3 - q^4 p_1 p_4 - \frac{1}{2}q^3 p_2^2 - q^4 p_2 p_3 + q^3/(q^4)^2, \\ H_{4,2}^{(-2)} &= -\frac{1}{2}q^2 p_1^2 - q^3 p_1 p_2 - q^4 p_1 p_3 + \frac{1}{2}(q^2)^2 p_2^2 + q^2 q^3 p_2 p_3 + q^2 q^4 p_2 p_4 - \frac{1}{2}q^1 q^3 p_2^2 \\ &\quad - p_2 q^1 q^4 p_3 - \frac{1}{2}q^4 p_2^2 + \frac{1}{2}(q^3)^2 p_3^2 + q^3 q^4 p_3 p_4 + \frac{1}{2}(q^4)^2 p_4^2 + q^1 q^3/(q^4)^2 - 1/q^4, \\ H_{4,3}^{(-2)} &= -\frac{1}{2}q^3 p_1^2 - q^4 p_1 p_2 + \frac{1}{2}q^3 q^2 p_2^2 + (q^3)^2 p_2 p_3 + q^3 q^4 p_2 p_4 - \frac{1}{2}q^1 q^4 p_2^2 + q^4 q^3 p_3^2 \\ &\quad + (q^4)^2 p_3 p_4 - q^1/q^4 + q^2 q^3/(q^4)^2, \\ H_{4,4}^{(-2)} &= -\frac{1}{2}q^4 p_1^2 + \frac{1}{2}q^4 q^2 p_2^2 + q^4 q^3 p_2 p_3 + (q^4)^2 p_2 p_4 + \frac{1}{2}(q^4)^2 p_3^2 - q^2/q^4 + (q^3)^2/(q^4)^2 \end{aligned}$$

are in involution by construction. Moreover, $H_{4,1}^{(-2)}$ commutes with $q^4 p_2$ and thus by Theorem 1 the quantities

$$F_{4,2}^{(-2)} = \frac{1}{2}q^4 p_1^2, \quad F_{4,3}^{(-2)} = \frac{1}{2}q^4 q^3 p_2^2 + (q^4)^2 p_2 p_3 - q^3/q^4, \quad F_{4,4}^{(-2)} = \frac{1}{2}(q^4)^2 p_2^2 + 1,$$

are additional integrals of motion. As $H_{4,1}^{(-2)}, H_{4,2}^{(-2)}, H_{4,3}^{(-2)}, H_{4,4}^{(-2)}, F_{4,2}^{(-2)}, F_{4,3}^{(-2)}, F_{4,4}^{(-2)}$ are functionally independent, the dynamical system associated with $H_{4,1}^{(-2)}$ is maximally superintegrable. Notice that all $(2n-1)$ integrals in this case are quadratic in momenta.

For all natural $n \geq 2$ we have $\partial V_1^{(m)}/\partial q^j = -1$ for $m = j$ and $\partial V_1^{(m)}/\partial q^j = 2q^1$ for $m = j+1$, whence $\{H_{i,1}^{(n-i)}, \frac{1}{2}p_{n-i}^2 - q^1\} = 0$ and $\{H_{i,1}^{(n-i+1)}, \frac{1}{2}p_{n-i}^2 + (q^1)^2\} = 0$ for $i = 0, \dots, n-1$. Therefore the dynamical systems associated with $H_{i,1}^{(n-i)}$ and $H_{i,1}^{(n-i+1)}$ for $i = 0, \dots, n-1$ possess $(n-1)$ additional integrals, namely $F_{i,s}^{(n-i)} = \{H_{i,s}^{(n-i)}, \frac{1}{2}p_{n-i}^2 - q^1\}$, $s = 2, \dots, n$, and $F_{i,s}^{(n-i+1)} = \{H_{i,s}^{(n-i+1)}, \frac{1}{2}p_{n-i}^2 + (q^1)^2\}$,

$s = 2, \dots, n$, respectively. However, in these cases the additional integrals are *cubic* in momenta, so we were unable to prove the functional independence of the sets $(H_{i,r}^{(k)}, r = 1, \dots, n, \text{ and } F_{i,s}^{(k)}, s = 2, \dots, n)$ for $i = 0, \dots, n-1$ and $k = n-i$ and $k = n-i+1$ in full generality so far. Nevertheless we are certain that these sets indeed are functionally independent for all these values of i and k , and hence the dynamical systems associated with $H_{i,1}^{(n-i)}$ and $H_{i,1}^{(n-i+1)}$ are maximally superintegrable.

For instance, let $n = 3, i = 0, k = 4$. The functions

$$\begin{aligned} H_{0,1}^{(4)} &= p_3 p_1 + \frac{1}{2} p_2^2 + q_1 p_2 p_3 + \frac{1}{2} p_3^2 q^2 + 2q^3 q^1 - 3q^2 (q^1)^2 + (q^2)^2 + (q_1)^4, \\ H_{0,2}^{(4)} &= p_1 p_2 + q_1 p_1 p_3 + q_1 p_2^2 + (q^1)^2 p_2 p_3 - \frac{1}{2} q^3 p_3^2 + \frac{1}{2} q^1 q^2 p_3^2 - (q^1)^2 q^3 + 2q^2 q^3 \\ &\quad - 2q^1 (q^2)^2 + (q^1)^3 q^2, \\ H_{0,3}^{(4)} &= \frac{1}{2} p_1^2 + q^1 p_1 p_2 + q^2 p_1 p_3 - q^3 p_2 p_3 + \frac{1}{2} (q^1)^2 p_2^2 + q^1 q^2 p_2 p_3 - \frac{1}{2} q^1 q^3 p_3^2 \\ &\quad + \frac{1}{2} (q^2)^2 p_3^2 + q^3 (-2q^2 q^1 + q^3 + (q^1)^3) \end{aligned}$$

again are in involution by construction; $H_{0,1}^{(4)}$ also commutes with $\frac{1}{2} p_3^2 + (q^1)^2$, and hence by the above so do

$$\begin{aligned} F_{0,2}^{(4)} &= -2q^1 p_2 - 3(q^1)^2 p_3 - \frac{1}{2} p_3^3 + 2q^2 p_3, \\ F_{0,3}^{(4)} &= -q^1 p_1 - 2(q^1)^2 p_2 - 4q^1 q^2 p_3 - p_2 p_3^2 - \frac{1}{2} q^1 p_3^2 + 2q^3 p_3 + (q^1)^3 p_3. \end{aligned}$$

As $H_{0,1}^{(4)}, H_{0,2}^{(4)}, H_{0,3}^{(4)}, F_{0,2}^{(4)}, F_{0,3}^{(4)}$ are functionally independent, the dynamical system associated with $H_{0,1}^{(4)}$ is maximally superintegrable as well. In contrast with the previous example, the additional integrals $F_{0,2}^{(4)}, F_{0,3}^{(4)}$ are *cubic* in momenta.

Acknowledgements

This research was partially supported by the Czech Grant Agency (GAČR) under grant No. 201/04/0538, Ministry of Education, Youth and Sports of the Czech Republic under grant MSM:J10/98:192400002 and the development project No. 254/b for the year 2004, KBN Research Grant No. 1 PO3B 111 27, and by the Silesian University in Opava through the internal grant IGS 1/2004. MB is pleased to acknowledge kind hospitality of the Mathematical Institute of Silesian University in Opava. The authors thank the anonymous referee for useful suggestions.

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